

Morphology of the two-dimensional MRI in Axial Symmetry

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In this paper, we analyze the linear stability of a stellar accretion disk, having a stratified morphology. The study is performed in the framework of ideal magneto-hydrodynamics and therefore it results in a characterization of the linear unstable magneto-rotational modes. The peculiarity of the present scenario consists of adopting the magnetic flux function as the basic dynamical variable. Such a representation of the dynamics allows to make account of the co-rotation theorem as a fundamental feature of the ideal plasma equilibrium, evaluating its impact on the perturbation evolution too. According to the Alfvénic nature of the Magneto-rotational instability, we consider an incompressible plasma profile and perturbations propagating along the background magnetic field. Furthermore, we develop a local perturbation analysis, around fiducial coordinates of the background configuration and dealing with very small scale of the linear dynamics in comparison to the background inhomogeneity size. The main issue of the present study is that the condition for the emergence of unstable modes is the same in the stratified plasma disk, as in the case of a thin configuration. Such a feature is the result of the cancelation of the vertical derivative of the disk angular frequency from the dispersion relation, which implies that only the radial profile of the differential rotation is responsible for the trigger of growing modes.

1. Introduction

The request for the existence of a linear unstable mode spectrum in a two-dimensional axially symmetric configuration clearly emerges from the theory of accretion on a compact astrophysical object. In fact, the Shakura idea Shakura (1973); Shakura & Sunyaev (1973) that the angular momentum transport across a stellar accretion disk is realized via an effective viscosity, naturally leads to search for a instability mechanism, able to trigger turbulence in the plasma profile. In the case of a differentially rotating disk, embedded in the central object gravitational field only, the axial symmetry of the configuration prevents that any linear unstable mode arises Bisnovatyi-Kogan & Lovelace (2001).

However, as a weak magnetic field is involved in the background disk morphology, the Velikhov-Chandrasekhar instability Velikhov (1959); Chandrasekhar (1960), also known as Magneto-rotational instability (MRI), is triggered. The basic mechanism underlying the emergence of such kind of instability is the direct coupling between the differential rotation of the disk and the magnetic field tension. By other words, the plasma disk inhomogeneity transform Alfvénic waves into growing modes.

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The detailed nature of the MRI and its role in triggering turbulence across an accretion disk has been extensively stated in Balbus & Hawley (1998). The condition for getting the emergence of growing modes is that the Alfvén frequency (the wavenumber amplitude times the Alfvén speed) is small compared with a frequency-like term containing the radial gradient of the disk angular velocity.

In Balbus (1995), the study of MRI is extended to a stratified disk (characterized by an inhomogeneous thick profile), outlining the role that the vertical derivative of the angular velocity plays in the morphology of the unstable modes. The study suggests that the driving force of the instability is to be identified in the spatial gradient of the angular velocity, differently from the non-magnetized case, where the dominant contribution comes from the specific angular momentum gradients. We stress how the analysis of the stratified disk, performed in Balbus (1995) relies on a vector formulation of the Magneto-hydrodynamics (MHD) ideal equations, i.e. no use is made of the magnetic flux surface function as a dynamical variable. Furthermore, this study does not rely on the validity of the co-rotation theorem, disregarded in the construction of the dispersion relation.

In the present paper, we perform a study of the stratified disk, similar to the one in Balbus (1995), but based on the magnetic surface function dynamics and directly accounting for the co-rotation theorem (the background disk angular frequency must depend on the flux surface Ferraro (1937)), holding for the background magnetic configuration. Without a significant loss of generality, we simplify the analysis of the dispersion relation for the linear mode spectrum, by considering a vanishing azimuthal component of the background magnetic field (as typically true for the magnetic field of a compact astrophysical object) and we also take perturbations which propagate along the background magnetic field only (this feature is intrinsic for Alfvénic disturbances of the background, as MRI results to be). More specifically, we deal with a local perturbation scheme, based on the construction of linear dynamics nearby fiducial background coordinates, as allowed by the assumption that the perturbation have a sufficiently short wavelength to explore a limited portion of the steady profile. Here, we consider an incompressible plasma at any order of approximation, coherently with the so-called Boussinesq approximation Balbus & Hawley (1998) (which states incompressible perturbations as a consequence of the mass conservation equation, when the large wavenumber hypothesis is implemented).

The main issue of our analysis is showing how the vertical derivative of the disk angular velocity cancels out from the dispersion relation, as far as the features of the co-rotation theorem are retained in the perturbation scheme. Indeed, our dispersion relation implies the same morphology of the MRI, (i.e. the same condition on the background parameters in order to trigger the instability), exactly like in the thin disk scenario.

This fact has a relevant physical implication for the accretion mechanism onto a compact object, since it states that only the radial differential rotation of the disk accounts for the instability property of the plasma and hence, only the radial steady disk profile really matters when the transport processes are analyzed.

The paper is organized as follows. In Sec. (2) we provide the general ideal MHD scheme to describe the plasma disk evolution in the formalism of the magnetic surface functions. All the basic equations are provided and the main implications of their structure are traced. In Sec. (3), we briefly describe the background plasma configuration, setting the basic force balance relations. In Sec. (4), we develop the linear perturbation equations, as written in the Fourier (plane wave) representation. The dispersion relation is then properly derived and the implication of its morphology are then discussed. Finally, in Section (5), brief concluding remarks follow.

2. Two-dimensional axisymmetric dynamics

We now provide the basic equations governing the two-dimensional axisymmetric dynamics of a plasma in the Magneto-hydrodynamical representation. Having in mind the specific application of such a dynamical system to the morphology and stability problem of an accretion disk, we write down the magnetic and velocity fields, making use of the magnetic flux surface ψ and the angular velocity ω , respectively, i.e.

$$\vec{B} = -\frac{1}{r}\partial_z\psi\hat{e}_r + \frac{\bar{B}_\phi}{r}\hat{e}_\phi + \frac{1}{r}\partial_r\psi\hat{e}_z \quad (2.1)$$

$$\vec{v} = \vec{v}_p + \omega r\hat{e}_\phi. \quad (2.2)$$

Above, $\vec{v}_p = v_r\hat{e}_r + v_z\hat{e}_z$ is the poloidal component of the velocity field and all the dynamical variables depend on t , r and z only (the ϕ dependence being suppressed because of the axial symmetry).

The dynamics of the two magnetic variables ψ and \bar{B}_ϕ can be easily fixed by the ideal MHD equations involving the magnetic structure of the plasma. In particular, the azimuthal component of the electron force balance provides the equation

$$\partial_t\psi + \vec{v}_p \cdot \vec{\nabla}\psi = 0. \quad (2.3)$$

Analogously, the azimuthal component of the induction equation yields the dynamics of \bar{B}_ϕ as

$$\partial_t\bar{B}_\phi + \vec{v}_p \cdot \vec{\nabla}\bar{B}_\phi + \bar{B}_\phi\vec{\nabla} \cdot \vec{v}_p = r(\partial_z\omega\partial_r\psi - \partial_r\omega\partial_z\psi). \quad (2.4)$$

The evolution of the angular velocity ω and of the poloidal component \vec{v}_p is described by the momentum conservation system, whose azimuthal component reads

$$\rho r\left(\partial_t\omega + \vec{v}_p \cdot \vec{\nabla}\omega\right) + 2\rho v_r\omega = \frac{1}{4\pi r^2}(\partial_r\psi\partial_z\bar{B}_\phi - \partial_z\psi\partial_r\bar{B}_\phi), \quad (2.5)$$

while the poloidal ones provide

$$\begin{aligned} \rho\left(\partial_t\vec{v}_p + \vec{v}_p \cdot \vec{\nabla}\vec{v}_p - \omega^2 r\hat{e}_r\right) = -\vec{\nabla}p - \\ -\frac{1}{4\pi r}\left[\partial_r\left(\frac{1}{r}\partial_r\psi\right) + \frac{1}{r}\partial_z^2\psi\right]\vec{\nabla}\psi - \\ -\frac{1}{8\pi r^2}\vec{\nabla}\bar{B}_\phi^2 + \vec{F}_p^e. \end{aligned} \quad (2.6)$$

Above \vec{F}_p^e is the external poloidal force acting in the disk and, in what follows, we will identify it with the gravity of the central astrophysical object.

The dynamics of the magnetized plasma, as described in the ideal MHD is then closed by providing the mass conservation relation, i.e. the continuity equation for the mass density ρ

$$\partial_t\rho + \vec{\nabla} \cdot (\rho\vec{v}_p) = 0 \quad (2.7)$$

and eventually assigning a suitable equation of state to express the pressure p , for instance in the barotropic form $p = p(\rho)$. In the perturbation analysis below, we will not need to specify the plasma equation of state, simply requiring its incompressibility.

We conclude by stressing how Eqs. (2.4) and (2.5) correlate the two azimuthal components of the magnetic and velocity fields respectively, making the gradient of one as source term for the generation of the other one (the left-hand-sides of these equations are linear homogeneous operators in the corresponding dynamical variable).

3. Background morphology

We consider as background plasma a purely rotating steady configuration $\vec{v}_0 = \omega_0 r \hat{e}_\phi$ (here the suffix 0 denotes background quantities). The rotation is clearly differential across the disk, i.e. $\omega_0 = \omega_0(r, z)$, but the validity of the co-rotation theorem Ferraro (1937) (holding for a stationary axisymmetric plasma) requires the condition $\omega_0 = \omega_0(\psi_0)$, ψ_0 being the background magnetic surface. Accounting for the validity of the co-rotation theorem is the basic feature of our study of the two-dimensional MRI and it constitute the major difference with the analysis in Balbus (1995).

When referred to the background equilibrium configuration, the dynamical system discussed in the previous section, reduces to a force balance system- The gravity of the central body, around which the plasma disk is orbiting, is crucial in fixing the steady morphology, via the gravostatic equilibrium equation, involving also the background mass density ρ_0 and pressure p_0 . Furthermore, since the plasma disk is embedded in the vacuum magnetic field of the central object, described via the function ψ_0 , the Lorentz force-free condition must hold. Thus, we respectively get the two equations

$$\vec{\nabla} p_0 = \rho_0 (\omega_0^2(\psi_0) r \hat{e}_r - \omega_K^2(r, z^2) \vec{r}_p) \quad (3.1)$$

$$\frac{1}{4\pi r} \left[\partial_r \left(\frac{1}{r} \partial_r \psi_0 \right) + \frac{1}{r} \partial_z^2 \psi_0 \right] = 0, \quad (3.2)$$

where ω_K denotes the Keplerian frequency and \vec{r}_p is the vector radius in the meridian plane.

After solving the second of these equations to get the form of ψ_0 , for instance in the form of a dipole contribution (typical in compact astrophysical objects), we can assign the function $\Omega_0(\psi)$ and the equation of state for the background plasma, to determine the profile of the mass density ρ_0 by the integration of the first one. However, for the perturbation analysis, here developed, such details are unessential (indeed we deal with a local approach) and we can directly proceed toward the linear dynamics. Finally, we note that, while for a thin configuration the disk angular velocity is almost Keplerian (i.e. $\omega_0 \simeq \omega_K$), the radial pressure gradient takes a relevant role for a thick disk profile (i.e. $\omega_0 \neq \omega_K$), see Ogilvie (1997).

4. Linear perturbation theory

We now address the linear perturbation approach, which is based on axisymmetric non-stationary corrections to the equilibrium, also assumed of very small spatial scale (their wavevectors have large magnitude) with respect to the background quantities. By other words, we deal with a local perturbation approach, in which the linear terms, denoted via the suffix 1, are expanded in Fourier series, analyzing the single monochromatic modes, i.e.

$$(\dots)_1(t, \vec{r}_p) = (\dots)_1 \exp \left\{ i \left(\vec{k} \cdot \vec{r}_p - \Omega t \right) \right\} \quad (\dots)_1 = \text{const.} \quad (4.1)$$

Treating the perturbations as local corrections ($|\vec{k} \cdot \vec{r}_0| \ll 1$, being r_0 the fiducial

radius, around which the mode lives), does not mean that we deal with a homogeneous background. In fact, each background quantity is calculated at the fiducial coordinates $\{r_0, z_0\}$ and it behaves as a constant coefficient in the perturbation scheme, but this is true also for the spatial gradients and, in particular, for the derivatives of the background angular velocity (which are expected to trigger the MRI).

Furthermore, we consider an incompressible plasma, for which $\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \vec{v}_p = 0$ and a vanishing background azimuthal magnetic field. Often (see Balbus & Hawley (1998)), such a request comes out as the so-called “Boussinesq approximation”, when a local approach is pursued, but here it must be regarded as a basic feature of the system, introduces to better select the Alfvénic signature of the MRI. This same point of view leads us to simplify our analysis, by eliminating the magnetic pressure with the request that the wavevector \vec{k} be parallel to the background magnetic field \vec{B}_0 , i.e. we require $\vec{k} \cdot \vec{\nabla} \psi_0 = 0$.

Since each monochromatic mode obeys the relations

$$\partial_t(\dots)_1 = -i\Omega(\dots)_1; \quad \vec{\nabla}(\dots)_1 = i\vec{k}(\dots)_1, \quad (4.2)$$

we can easily restate the dynamics of the perturbations (as deduced by the basic system of the ideal MHD evolution equations) in terms of a closed algebraic system, providing the dispersion relation for the mode spectrum.

We observe how the poloidal velocity \vec{v}_p is absent in the background equilibrium and it is therefore natural to express it by means of the poloidal shift vector $\vec{\xi}_p$ of the plasma elements, i.e. we have

$$\vec{v}_p = \partial_t \vec{\xi}_p = -i\Omega \vec{\xi}_p \Rightarrow \vec{\nabla} \cdot \vec{\xi}_p = i\vec{k} \cdot \vec{\xi}_p = 0. \quad (4.3)$$

First of all, we stress how the continuity equation (2.7) is reduced by the incompressibility constraint above to the vanishing nature of the perturbed mass density ρ_1 . Furthermore, Eq. (2.3) provides the perturbed magnetic surface function ψ_1 in the form

$$\psi_1 = -\vec{\xi}_p \cdot \vec{\nabla} \psi_0. \quad (4.4)$$

Hence, the poloidal component (2.6) of the momentum conservation system reads, to the linear approximation, as follows

$$\Omega^2 \vec{\xi}_p + 2\omega_0 (\dot{\omega}_0 \psi_1 + \omega_1^+) \hat{e}_r = i\vec{k} \frac{p_1}{\rho_0} - \frac{k^2 \psi_1}{4\pi r^2 \rho_0} \vec{\nabla} \psi_0. \quad (4.5)$$

Above, we split the perturbed angular velocity in terms of its first order co-rotation contribution $\dot{\omega}_0 \psi_1$ (where $\dot{\omega}_0 \equiv (d\omega/d\psi)_{\psi=\psi_0}$) and a generic linear deformation ω_1^+ .

Taking the scalar product of the system above with the wavevector \vec{k} , we easily get the perturbed pressure contribution as a consequence of preserving the incompressibility condition during the dynamics, i.e.

$$k^2 p_1 = -2i\rho_0 \omega_0 k_r r (\dot{\omega}_0 \psi_1 + \omega_1^+). \quad (4.6)$$

The scalar product of Eq. (4.5) with $\vec{\nabla} \psi_0$, once Eq. (4.4) is used, yields the basic relation

$$(\Omega^2 - y_r - \omega_A^2) \psi_1 = 2\omega_0 r \partial_r \psi_0 \omega_1^+, \quad (4.7)$$

where $\omega_A^2 \equiv (\vec{k} \cdot \vec{B}_0)^2 / 4\pi \rho_0 = k^2 v_A^2$ (v_A being the Alfvén velocity) and $y_r \equiv 2\omega_0 r \partial_r \omega_0$.

For a monochromatic mode the linearized Eqs. (2.4) and (2.5) take respectively the form

$$r\Omega\omega_1^+ = -2\omega_0\Omega\xi_r - \frac{\vec{k}\cdot\vec{B}_0}{4\pi\rho_0 r}(\bar{B}_\phi)_1 \quad (4.8)$$

and

$$\Omega(\bar{B}_\phi)_1 = -r^2\vec{k}\cdot\vec{B}_0\omega_1^+, \quad (4.9)$$

which combined together give the relation

$$r(\Omega^2 - \bar{\omega}_A^2)\omega_1^+ = -2\omega_0\Omega^2\xi_r. \quad (4.10)$$

Finally, the radial component of Eq. (4.5), using Eq. (4.6), can be restated as

$$\Omega^2\partial_r\psi_0\xi_r = -(\alpha y_r + v_{Az}^2 k^2)\psi_1 - 2\alpha\omega_0 r\partial_r\psi_0\omega_1^+, \quad (4.11)$$

where $\alpha \equiv 1 - k_r^2/k^2$.

The system of Eqs. (4.7), (4.10) and (4.11) is a closed algebraic set in the variables ψ_1 , ω_1^+ and ξ_r and they can be easily combined to obtain the following dispersion relation

$$\begin{aligned} \Omega^4 - b\Omega^2 + c &= 0, \\ c &\equiv \omega_A^2(y_r + \omega_A^2) \quad b \equiv (K_0^2 + 2\omega_A^2) + 4\omega_0^2(\alpha - 1), \end{aligned} \quad (4.12)$$

Here

$$K_0^2 \equiv \frac{1}{r^3}\partial_r(r^4\omega_0^2) = y_r + 4\omega_0^2. \quad (4.13)$$

It is immediate to check that the necessary condition to get MRI is provided by the inequality $\omega_A^2 < -y_r$, which ensures $c < 0$. Indeed, also the request $b < 0$ could provide unstable behaviors, but the expression of K_0^2 implies that $b = c/\omega_A^2 + 4\omega_0^2(k_z^2/k^2)$ and hence $b < 0$ requires again $c < 0$.

Thus, we see how, when we take into account the validity of the co-rotation theorem, the condition for MRI is the same in the case of a stratified thick disk as in a thin disk configuration Balbus & Hawley (1991). Such an issue relies on the cancellation of the z -derivative of the angular velocity in the dispersion relation and this fact is the main difference between the present analysis and that one performed in Balbus (1995), where a vector formulation is addressed instead using the magnetic flux surface function (regardless of the co-rotation theorem).

5. Concluding remarks

We analyzed the stability of a stratified and differently rotating ideal plasma disk, as described in a two-dimensional axisymmetric scheme. After the set up of the basic dynamical system, required to address the considered problem, we briefly characterize the morphology of a the steady background configuration of the plasma disk. The steady configuration is assumed to be embedded in the gravitational and the magnetic field of the central object, according to the astrophysical implementation of our analysis in the behavior and stability of stellar accretion structures. Then, we constructed the Fourier representation of the linear perturbation dynamics, allowed by the request to deal with

short wave-length deformation of the background and hence by a local perturbation approach. The combination of such algebraic equations lead to the morphology of the dispersion relation, providing the structure of the spectral modes. As result of such a procedure, we arrive to study the profile of the MRI in the case of a stratified disk, evaluated with the help of two simplifying assumptions for selecting the Alfvénic nature of this instability: the plasma is required to be incompressible (so removing the acoustic modes) and the perturbations propagate along the background magnetic field (i.e. we deal with zero first order magnetic pressure). As basic issue of our analysis we demonstrate how, accounting for the co-rotation theorem on the background, has significant implications on the linear dispersion relation. In particular, the co-rotation theorem is at the ground of the cancelation of the vertical derivative of the background disk angular velocity from the structure of the linear unstable modes. Such a feature does not emerge in the vector approach depicted in Balbus (1995), where the co-rotation profile is not taken into account. Thus, the dispersion relation for a stratified thick accretion disk retains the same structure as in the thin disk approximation, i.e. the condition to trigger the MRI is associated to the same inequality. As a result the stability of the disk, the corresponding turbulence and the associated angular momentum transport are essentially determined by the radial profile of the background configuration, which is fixed by the balance between the gravitational force, the pressure gradient and the centripetal force. The main difference in the MRI morphology between thin and thick accretion disks, consists in the role that the radial pressure can take in the later case, while in the former the disk angular frequency has essentially a Keplerian profile.

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